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APPLICATIONS OF NATURAL TRANSFORM IN DISTRIBUTIONS

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ABSTRACT
In this paper,a new integral transform, Natural transform is introduced in a generalized sense(Distributional Space)and presented its applications in a distribution space.

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1 INTRODUCTION

The scientific disciplines like physics,chemistry,mathematics,biology comes under one roof called as basic sciences.In basic sciences we study the natural phenomenon which occurs in above disciplines.Among all these disciplines mathematics plays the vital role due to its interdisciplinary approach.As a result of that,in the development of interdisciplinary research application of mathematics has significant role.There are fields like physics, engineering etc. where the mathematical methods and tools are being used to resolve the problems encoun- tered in the study of these subject.This shows that mathematics has a great contribution in the development of basic sciences.

The concept of function is one of the important aspect in the subject of mathematics.To define a function one need to specify the proper domain and co-domain by means of which the said function is well defined.In the growing the research,a function can be viewed by its action as a functional on the given testing space.This leads to introduction of new concept called as gener- alized functions.Now a days,the theory of generalized functions becomes the growing branch of pure and applied mathematics and attract the researcher due to its wide range of applica- tions.In this theory of generalized functions,a function is governed by its action as a functional on the given testing space.With the help of these generalized functions one can solve the problems of ordinary differential equations,partial differential equations,integral equations,fluid mechanics etc.The aim of this paper is to define the new integral transform Natural transform on the given distribution space and to study of this generalized Natural transform to various fields.

THE NATURAL TRANSFORMATION

Recently,the new integral transform Natural transform(N-transform) was introduced by Khan and Khan [12] and studied its properties and some applications.The Natural transform of the function $f(t) \in R^2$ is denoted by symbol $N[f(t)] = R(s,u)$ where s and u are the transform variables and is defined by an integral equation[1]

$$N[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)\,dt \tag{1.1}$$

where $Re(s) > 0$, $u \in (\tau_1, \tau_2)$,the function $f(t)$ is sectionwise continuous,exponential order and defined over the set

$A = [f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty)]$.

The above equation can be written in another form as

$$N[f(t)] = R(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)\,dt \tag{1.2}$$

The inverse Natural transform of function $R(s,u)$ is denoted by symbol $N^{-1}[R(s,u)] = f(t)$ and is defined with Bromwich contour integral[1,2]

$$N^{-1}[R(s,u)] = f(t) = \lim_{T \to \infty} \frac{1}{2\Pi i} \int_{\gamma - iT}^{\gamma + iT} e^{\frac{st}{u}} R(s,u)\,ds \tag{1.3}$$

We can extract the Laplace, Sumudu, Fourier and Mellin transform from Natural transform and which shows that Natural transform convergence to Laplace and Sumudu transform[11]. Moreover Natural transform plays as a source for other transform and it is the theoretical dual of Laplace transform.Further study and applications of Natural transform can be seen in[3,4,5,6,7,8].In [9,10] we can see that, authors have presented the solution of distributional Abel integral equation by distributional Sumudu transform and Fractional Integrals and Derivatives for Sumudu transform on distribution spaces.

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Some Preliminary Results of Natural Transform

(1) $N[1] = \frac{1}{s}$ (2) $N[t^n] = \frac{u^n}{s^{n+1}} n!$ (3) $N[e^{at}] = \frac{1}{s - au}$

(4) $N\left[\frac{t^{n-1}e^{at}}{(n-1)!}\right] = \frac{u^{n-1}}{(s - au)^2}$ (5) $N[f^{(n)}(t)] = \frac{s^n}{u^n}.R(s, u) - \sum_{n=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0)$

The Heaviside Function

The Heaviside function $H(x)$ assign the value zero for every negative value of x and assign unity for every positive value of x, that is,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

It has a jump discontinuity at $x = 0$ and is also called the unit step function. Its value at $x = 0$ is usually taken to be $\frac{1}{2}$. Sometimes it is taken to be a constant c, $0 < c < 1$, and then the function is written $H_c(x)$. If the jump in the Heaviside function is at a point $x = a$, then the function is written H_{x-a}.

$\therefore H(-x) = 1 - H(x),\ H(a - x) = 1 - H(x - a)$

The function $H(x)$ plays an important role in describing the functions which are having jump discontinuity and in the study of generalized functions. Let $F(x)$ be a function which is continuous everywhere except for the point $x = \xi$, at which $F(x)$ has a jump discontinuity

$$F(x) = \begin{cases} F_1(x) & x < \xi \\ F_2(x) & x > \xi \end{cases}$$

From this equation, we can write $F(x)$ as

$F(x) = F_1(x)H(\zeta - x) + F_2(x)H(x - \zeta)$

This concept can be generalized for more than one point if jump discontinuity.

The Dirac Delta Function

In physical problem we often encounters idealized concepts such as a force concentrated at a point or an impulsive force that acts instantaneously. These forces are described by the Dirac delta function $\delta(x - \xi)$ which has several significant properties.

$$\delta(x - \xi) = 0,\ x \neq \xi \tag{1.4}$$

$$\int_a^b \delta(x - \xi) = \begin{cases} 0 & a, b < \xi \text{ or } \xi < a, b \\ 1 & a \le \xi \le b \end{cases} \tag{1.5}$$

$$\int_{-\infty}^{\infty} \delta(x - \xi)dx = 1 \tag{1.6}$$

$$\int_{-\infty}^{\infty} \delta(x - \xi)f(x)dx = f(\xi) \tag{1.7}$$

Where $f(x)$ is a sufficiently smooth function, equation (1.7) is called shifting property or the reproducing property of the delta function. The language of classical mathematics is inadequate to justify such function. In a classical mathematics, if a function is zero except at a point then its integral is necessarily zero, without regard for the definition used for the integral, which shows contradiction to equation (1.4) and (1.5). But there are some sequences which shows the property (1.7) i.e. $\lim_{m \to \infty} \int_{-\infty}^{\infty} f(x) \frac{\sin(mx)}{\pi x} dx = f(0)$. This is called as Dirichlet Formula.

237

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2 Generalized Natural Transform

Testing Function Space $\mathbf{D}_{a,b}$

Let $\mathbf{D}_{a,b}$ denotes the space of all complex valued smooth functions $\varphi(t)$ on $-\infty < t < \infty$ on which the functions $\gamma_k(\varphi)$ defined by

$\gamma_k(\varphi)$ ¾ $\gamma_{a.b.k}(\varphi)$ ¾ $\sup\limits_{0<t<\infty} |K_{a,b}(t)D^k(t)$ ⊧ ∞ (2.1)

Where

$$K_{a.b}(t) = \begin{cases} e^{at} & 0 \le t < \infty \\ e^{bt} & -\infty < t < 0. \end{cases}$$

This $\mathbf{D}_{a,b}$ is linear space under the pointwise addition of function and their multiplication by complex numbers.Each γ_k is clearly a seminorm on $\mathbf{D}_{a,b}$ and γ_0 is a norm.We assign the topology generated by the sequence of seminorm $(\gamma_k)_{k=0}^{\infty}$ there by making it a countably multinormed space.Note that for each fixed s and u the kernel $\frac{1}{u}e^{\frac{-st}{u}}$ as a function of t is a member of $\mathbf{D}_{a,b}$ iff $a < Re(\frac{s}{u}) < b$.With the usual argument we can show that $\mathbf{D}_{a,b}$ is complete and hence a Frechet space.$\mathbf{D}'_{a,b}$ denotes the dual of $\mathbf{D}_{a,b}$ i.e. f is member of $\mathbf{D}'_{a,b}$ iff it is continuous linear function on $\mathbf{D}_{a,b}$.Thus $\mathbf{D}'_{a,b}$ is a space of generalized functions.

Now we define the generalized Natural Transform.Given a generalized Natural transformable generalized function f,the strip of definition Ω_f for N[f] is a set in C defined by Ω_f ¾ $\{(s,u)$: $\omega_1 < Re(\frac{s}{u}) < \omega_2\}$ since for each $(s,u) \in \Omega_f$ the kernel $e^{\frac{-st}{u}}$ as a function of t is a member of $\mathbf{D}'_{\omega_1,\omega_2}$.

For $f \in \mathbf{D}'_{\omega_1,\omega_2}$ we can define the generalized Natural transform of f as conventional function

$R_f(s,u)$ ¾ N$[f(t)]$ ¾ $< f(t), \frac{1}{u}e^{\frac{-st}{u}} >$ (2.2)

We call Ω_f the region (or strip) of definition for N$[f(t)]$ and ω_1 and ω_2 the abscissas of definition. Note that the properties like linearity and continuity of generalized Natural transform will follows from [13,14]

3 Natural Transform of Distribution

Let $f(t)$ be a distribution whose support is bounded on the left at zero,then the Natural transform of $f(t)$ is defined by equation

$$\mathrm{N}[f(t)] = \bar{f}(s,u) = \int_0^{\infty} f(t)e^{\frac{-st}{u}}dt =< f(t), \frac{1}{u}e^{\frac{-st}{u}} > \quad (3.1)$$

Here we can examine the above relation in a way that, there exists a real number $\frac{c_1}{c_2}$ such that $e^{\frac{c_1}{c_2}t}$ is a distribution belonging to S'(the class of tempered distribution).Then we can write equation (5) as

$$\mathrm{N}[f(t)] = R_f(s,u) = \int_0^{\infty} f(t)e^{\frac{-st}{u}}dt =< e^{\frac{-c_1}{c_2}t}f(t), H(t)\frac{1}{u}e^{\frac{-(s-c)t}{u}} > \quad (3.2)$$

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where $H(t)$ is the Heaviside function. For $Re(\frac{s}{u}) > Re(\frac{c}{u})$, the function $H(t)\frac{1}{u}e^{\frac{-(s-c)t}{u}}$ is a test function in S, and hence the above definition makes sense.

4 The Natural Transform of Heaviside Function and Dirac Delta Function

1 The Heaviside Function $N[H(t)] = \int_0^\infty e^{\frac{-st}{u}} = \frac{u}{s}$

2 The Dirac Delta Function and its Derivatives

$$N[\delta(t-c)] = \frac{1}{u}e^{\frac{-cs}{u}} \qquad N[\delta(t)f(t)] = \frac{1}{u}f(c)e^{\frac{-cs}{u}} \qquad N[\delta'(t-c)] = \frac{s}{u^2}e^{\frac{-cs}{u}}$$

In general, $N[\delta^{(n)}(t-c)] = (\frac{s^n}{u^{n+1}})e^{\frac{-cs}{u}}$

If we set $c = 0$, then we have

$$N[\delta(t)] = \frac{1}{u} \qquad N[\delta(t)f(t)] = \frac{1}{u}f(o)$$
$$N[\delta'(t)] = \frac{s}{u^2}$$

In general, $N[\delta^{(n)}(t)] = (\frac{s^n}{u^{n+1}})$

5 The Natural Transform of Distributional Derivative

Let $f(t) \in S'$, then by the definition of Natural transform we have

$$N[\overline{f'(t)}] = < \overline{f'(t)}, \frac{1}{u}e^{\frac{-st}{u}} > \tag{5.1}$$

Let $f(t)$ be a function defined by

$$f(t) = \begin{cases} g_1(t) & t < a \\ g_2(t) & t > a \end{cases} = g_1(t)H(a-t) + g_2(t)H(t-a) \tag{5.2}$$

where $a > 0$ and $g_1(t)$, $g_2(t)$ are continuously differentiable functions. The classical derivative of $f(t)$ is given by

$$f'(t) = g_1'(t)H(a-t) + g_2'(t)H(t-a) \tag{5.3}$$

For all $t \neq a$, the distributional derivative is

$$\overline{f'(t)} = f'(t) + [f]\delta(t-a) \tag{5.4}$$

where $[f] = f(a_+) - f(a_-)$

The Natural transform of equation (5.3) is given by

$$N[f'(t)] = \frac{1}{u}\int_0^\infty f'(t)e^{\frac{-st}{u}} = \frac{1}{u}\int_0^a g_1'(t)e^{\frac{-st}{u}} + \frac{1}{u}\int_a^\infty g_2'(t)e^{\frac{-st}{u}}$$
$$= \frac{1}{u}\{[e^{\frac{-st}{u}}g_1(t)]_0^a + \frac{s}{u}\int_0^a g_1(t)e^{\frac{-st}{u}}\} + \frac{1}{u}\{[e^{\frac{-st}{u}}g_2(t)]_a^\infty + \frac{s}{u}\int_a^\infty g_2(t)e^{\frac{-st}{u}}\}$$
$$= \frac{s}{u^2}[\int_0^a g_1(t)e^{\frac{-st}{u}} + \int_a^\infty g_2(t)e^{\frac{-st}{u}}] - \frac{1}{u}(e^{\frac{-as}{u}}[f] + g_1(0))$$
$$= \frac{s}{u^2}f(t) - \frac{1}{u}(f(0) + [f]e^{\frac{-as}{u}})$$

239

 ISSN 2394 - 7780

where $f(0) = g_1(0)$

On the other hand

$$N[\overline{f'(t)}] = < \overline{f'(t)}, e^{\frac{-st}{u}} >$$
$$= < f'(t) + [f]\delta(t-a), \frac{1}{u} e^{\frac{-st}{u}} >$$
$$= \frac{1}{u}\int_0^\infty f'(t) e^{\frac{-st}{u}} + \frac{1}{u}[f] e^{\frac{-as}{u}}$$
$$= \frac{s}{u^2} f(\bar{t}) - \frac{1}{u} f(0)$$

The above relation makes the sense even when we allow a to tend to zero, because in that case we have

$$N[f'(t)] = \frac{1}{u}\int_0^\infty g'_2(t) e^{\frac{-st}{u}}$$
$$= \frac{s}{u^2}\int_0^\infty g(t) e^{-st} dt - g(0)$$
$$= \frac{s}{u^2} f(\bar{t}) - \frac{1}{u} f(0+)$$
$$= \frac{s}{u^2} f(\bar{t}) - \frac{1}{u} f(0_-) - [f(0_+) - f(0_-)] e^{\frac{-0s}{u}}$$

which is consistent with above equation.

6 Application of Generalized Natural Transform

Example (1) Solve $y'' + 6y' + 5y = \delta(t) + \delta(t-2)$ with initial condition $y(0) = 1$, $y'(0) = 0$

Solution:- Taking Natural transform on both sides given equation and simplifying

$$N[y'' + 6y' + 5y] = N[\delta(t) + \delta(t-2)] \tag{6.1}$$

$$[\frac{s^2}{u^2}N[y] - \frac{s}{u^2}y(0) + \frac{1}{u}y'(0)] + 6[\frac{s}{u}N[y] - \frac{1}{u}y(0)] + 5N[y] = \frac{1}{u} + e^{\frac{-2s}{u}} \tag{6.2}$$

$$N[y][\frac{s^2 + 6su + 5u^2}{u^2}] = \frac{s}{u^2} + \frac{6}{u} + \frac{1}{u} + \frac{1}{u}e^{\frac{-2s}{u}} \tag{6.3}$$

$$N[y] = \frac{s + 7u}{s^2 + 6su + 5u^2} + \frac{u}{s^2 + 6su + 5u^2} e^{\frac{-2s}{u}} \tag{6.4}$$

Now using the partial fraction method we have

$$N[y] = \frac{\frac{3}{2}}{s+u} + \frac{\frac{-1}{2}}{s+5u} + (\frac{\frac{1}{4}}{s+u} - \frac{\frac{1}{4}}{s+5u}) e^{\frac{-2s}{u}} \tag{6.5}$$

Applying inverse Natural transform on both sides we get

$$y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-5t} + \frac{1}{4}u_2(t)(e^{-t+2} - e^{-5t+10}) \tag{6.6}$$

Example (2) Solve $y'' + 2y' + 10y = -\delta(t - 4\pi)$ with initial condition $y(0) = 0$, $y'(0) = 1$

240

 ISSN 2394 - 7780

Solution: Taking Natural transform on both sides given equation and simplifying

$$N[y'' + 2y' + 10y] = N[\delta(t - 4\pi)] \tag{6.7}$$

$$[\frac{s^2}{u^2}N[y] - \frac{s}{u^2}y(0) + \frac{1}{u}y'(0)] + 2[\frac{s}{u}N[y] - \frac{1}{u}y(0)] + 10N[y] = \frac{1}{u}e^{\frac{-4\pi s}{u}} \tag{6.8}$$

$$N[y][\frac{s^2 + 2su + 10u^2}{u^2}] = -\frac{1}{u} - \frac{1}{u}e^{\frac{-4\pi s}{u}} \tag{6.9}$$

$$N[y] = \frac{u}{s^2 + 2su + 10u^2} + \frac{u}{s^2 + 2su + 10u^2}e^{\frac{-4\pi s}{u}} \tag{6.10}$$

Applying inverse Natural transform on both sides we get

$$N^{-1}[y] = N^{-1}[\frac{u}{s^2 + 2su + 10u^2} + \frac{u}{s^2 + 2su + 10u^2}e^{\frac{-4\pi s}{u}}] \tag{6.11}$$

The expression $\frac{u}{s^2+2su+10u^2}$ can be written as $\frac{1}{3}[\frac{3u}{(s+u)^2+9u^2}]$ whose inverse Natural transform found to be $\frac{1}{3}e^{-t}Sin(3t)$.Also in second part ,the factor $e^{\frac{-4\pi s}{u}}$ gains its effect of a unit step function and a translation by $c = 4\pi$ so that its inverse Natural transform is $-\frac{1}{3}u_{4\pi}(t)(e^{-(t-4\pi)})$

$$y(t) = \frac{1}{3}e^{-t}Sin(3t) - \frac{1}{3}u_{4\pi}(t)e^{-(t-4\pi)}$$

$$= \frac{1}{3}e^{-t}Sin(3t) - \frac{1}{3}u_{4\pi}(t)e^{-t+4\pi}$$

Example (3) A simply supported beam of length L bars a load P concentrated at its midpoint($x = \frac{L}{2}$).Find the deflection of the beam.

Solution: We know the equation of beam with the required boundary conditions as

$$EIy^{(4)} = P\delta(x - \frac{L}{2}) \tag{6.12}$$

with, $y(0) = y''(0) = y(L) = y''(L) = 0$

Assume that the function y(x)is defined on the domain $[0, \infty)$,but it only has physicalmeaning on $[0, L]$.From the initial conditions given we can see that, the conditions like $y'(0)$ and $y''(0)$ are not known.To solve the given problem ,we can assign some value to these conditions for our convenience say $y'(0) = a$ and $y''(0) = b$,which we can determine using the known conditions.

Now apply the Natural transform on both sides and using the initial conditions,we get

$$EIN[y^{(4)}] = PN[\delta(x - \frac{L}{2})]$$

$$EI[\frac{s^4}{u^4}N[y] - \frac{s^3}{u^4}y(0) - \frac{s^2}{u^3}y'(0) - \frac{s}{u^2}y''(0) - \frac{1}{u}y'''(0)] = P\frac{1}{u}e^{\frac{-sL}{2u}}$$

$$EI\,N[y] = \frac{u}{s^2}a + \frac{u^3}{s^4}b + \frac{Pu^3}{s^4}\frac{1}{EI}e^{\frac{-sL}{2u}}$$

Now apply the inverse Natural transform ,we have

$$y(x) = ax + b\frac{1}{6}x^3 + P\frac{1}{6EI}u_{(\frac{L}{2})}(x)(x - \frac{L}{2})^3 \tag{6.13}$$

Now using the boundary conditions $y(L) = 0$ and $y''(L) = 0$,we get

$$y''(x) = bx + P\frac{1}{EI}(x - \frac{L}{2}) \tag{6.14}$$

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$\therefore y(L) = aL + b\frac{1}{6}L^3 + P\frac{L^3}{48EI} = 0$ and $y'(L) = bL + P\frac{l}{2EI}$

Solving these equations for a and b ,we get

$a = \frac{PL^2}{16EL}$ and $b = \frac{-P}{2EI}$

Finally the deflection curve is given by

$$y(x) = \frac{PL^2}{16EL}x + \frac{-P}{2EI}\frac{1}{6}x^3 + P\frac{1}{6EI}u_{(\frac{L}{2})}(x)(x - \frac{L}{2})^3 \qquad (6.15)$$

More explicitly, the beam deflection is given by

$$y(x) = \begin{cases} \frac{PL^2}{16EL}x + \frac{-P}{12EI}x^3 & 0 \leq x < \frac{L}{2} \\ \frac{PL^2}{16EL}x + \frac{-P}{12EI}x^3 + \frac{P}{6EI}u_{(\frac{L}{2})}(x)(x - \frac{L}{2})^3 & \frac{L}{2} \leq x \leq L. \end{cases}$$

7 CONCLUSION

In this paper,we have defined the Natural transform on the distribution space with some suitable support.The Natural transform of the generalized functions like Dirac delta function,Heaviside functions are found and solved some differential equations which involves these generalized functions.

REFERENCES

[1] Belgacem FBM,Silambarasan R.Theory of the Natural transform.Mathematics in Engi- neering Science and Aerospace(MESA)journal.3(1),99-124,2012

[2] Belgacem FBM,Silambarasan R.Advances in the Natural transform.9th International Con- ference on Mathematical Problems in Engineering.Aerospace and Sciences AIP Confer- ence Proceeding.1493,106-110,2012

[3] Belgacem FBM,Silambarasan R.Applications of the Natural transform to Maxwell's Equa- tions.PIERS Suzhou,China.899-902,2011

[4] Chindhe A.D.,Kiwne S.B.Application of Combine Natural Transform and Adomian Decom- position Method in Volterra Integro-Differential Equations.Mathematical Journal of Inter- disciplinary Sciences.5(1),1–14,2016

[5] Chindhe A.D.,Kiwne S.B.Natural Transform and Special Functions.Journal of New Theory.15,39-47,2017

[6] Chindhe A.D.,Kiwne S.B.On The Generalized Natural Transform.Mathematical Journal of Interdisciplinary Sciences.5(1),1–14,2018

[7] Deshna Loonker,Banerji PK. Natural transform for distribution and Boehmian spaces,Mathematics in Engineering,Science and Aerospace(MESA),Vol. 4, No. 1, pp. 69- 76, 2013

[8] Deshna Loonker,Banerji PK.Application of Natural transform to differential equations. Journal Indian Academy of Mathematics.35(1),151-158,2013

[9] Deshna Loonker,Banerji PK. On the solution of distributional Abel integral equation by distributional Sumudu transform, Internat. J. Math. Math. Sci. 2011 (2011), 1-8, Article ID 480528.

[10] Deshna Loonker,Banerji PK. Fractional Integrals and Derivatives for Sumudu transform on distribution spaces,Appl. Appl. Math. Internat. J. 7 (1) (2012), 188-200

[11] Kamal Shah, Muhammad Junaid, Nigar Ali.Extraction of Laplace, Sumudu, Fourier and Mellin Transform from the Natural Transform,J. Appl. Environ. Biol. Sci.,5(9),108–115, 2015.

[12] Khan ZH,Khan WA.N-transform properties and applications.NUST Journal of Engineering Sciences.1(1),127-133,2008

[13] Zemanian, A. H. Distribution Theory and Transform Analysis : An Introduction to General- ized Functions, with Applications,Mc-Graw Hill, New York, (1965), Republished by Dover Publication, Inc., New York (1987).

[14] Zemanian, A. H. Generalized Integral Transformations, Interscience Publishers, New York (1968), Republished by Dover Publications, Inc., New York (1987).

242